5. iii) $\sum_{n=1}^{\infty} ne^{-2n}$

we consider the function $f(x) = xe^{-x^2}$

- Its positive and continuous in the interval $[1, \infty]$.

By examining its derivative Hr decreasing.

$$f'(x) = e^{-x^2} - 2e^{-x^2}x^2$$

$$= e^{-x^2}(1-2x^2) \quad < 0 \text{ if } x \geq 1$$

since $f(n) = ne^{-n^2}$

thus $\sum_{n=1}^{\infty}$. It suffices to check what happens to $\int_1^{\infty} f(x)\,dx$

$$\to \int_1^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}dx$$

$$= \frac{1}{-2}\lim_{t\to\infty}\int_{-1}^{-t^2} e^u\,du \quad * \; u = -x^2$$

$$= \lim_{t\to\infty} \frac{-1}{2} e^u \Big|_{-1}^{-t^2} = \lim_{t\to\infty} \frac{-1}{2}\left[e^{-t^2} - e^{-1}\right] = \frac{1}{2e}$$

thus integral converges)

Hence series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

Drop the absolute value bars.

$\Rightarrow$ Integral diverges and hence the Series $\sum_{n=1}^{\infty} \frac{1}{n \ln(n+1)}$

diverges.

8). $f(x) = \frac{1}{x \ln}$

$f(x) = \frac{1}{x \ln(x+1)}$, So we are going to eliminate 1st term and do the integral.

$\int \frac{1}{x \ln(x+1)}$ => non-negative.

$\lim_{n \to \infty} \int_1^{\infty} \frac{1}{x} \frac{1}{\ln(x+1)}$ Evaluating $\int_1^{\infty} \frac{1}{x \ln(x+1)} dx$

anti-derivative for $\int \frac{dx}{x \ln(x+1)}$.

Let $u = \ln(x+1)$, $du = \frac{2}{x+1} dx = dx/x$.

$\therefore \int \frac{du}{u} = \ln|u| = \ln|\ln(x+1)|$

$\Rightarrow \int_1^{\infty} \frac{dx}{x \ln(x+1)} = \lim_{t \to \infty} \int_1^t \frac{dx}{x \ln(x+1)} = \lim_{t \to \infty} \left(\ln|\ln(x+1)| \right) \Big|_1^t$

Evaluate the limit. $(\ln(\ln(t+1)) - \ln(\ln 2)) = \infty - \ln(\ln 2)$

$\therefore \cos 2x = \cos \cos$

5(vi) $\sum_{n=0}^{\infty} \frac{5n^2+n+1}{n^3+2}$ $n = 0, 1, 2, 3 \ldots$

$S_n = \sum_{n=0}^{\infty} \frac{5n^2+n+1}{n^3+2} + \sum_{n=1}^{\infty} \frac{5n^2+n+1}{n^3+2} + \sum_{n=2}^{\infty} \frac{5n^2+n+1}{n^3+2} + \ldots$

$= \frac{5(0)^2+0+1}{0^3+2} + \frac{5(1)+1+1}{1^3+2} + \frac{5(2)^2+2+1}{2^3+2}$

$= \frac{1}{2} + \frac{7}{3} + \frac{23}{10}$

$= 0.5 + 2.33 + 2.3$

$= 5.133 > 1$

Thus its divergent.

Thus the series $\sum_{n=0}^{\infty} \frac{5n^2+n+1}{n^3+2}$ diverges.

5(i) $\sum_{n=0}^{\infty} \frac{\pi^2}{9^n}$

Taking $n = 0, 1, 2, 3 \ldots$

$S_n = \sum_{n=0}^{\infty} \frac{\pi^2}{9^n} + \sum_{n=1}^{\infty} \frac{\pi^2}{9^n} + \sum_{n=2}^{\infty} \frac{\pi^2}{9^n} + \ldots$

Thus $\sum_{n=0}^{\infty} \frac{\pi^2}{9^0} + \sum_{n=1}^{\infty} \frac{\pi^2}{9^1} + \sum_{n=2}^{\infty} \frac{\pi^2}{9^2} + \ldots$

$\frac{\pi^2}{1} + \frac{\pi^2}{9} + \frac{\pi^2}{18} + \ldots$

$\frac{\pi^2}{1} + \frac{\pi^2}{9} + \frac{\pi^2}{18} = \frac{18\pi^2 + 2\pi^2 + \pi^2}{18}$

$= \frac{21\pi^2}{18} = \frac{7}{6}\pi^2 > 1$

Thus $\sum_{n=0}^{\infty} \frac{\pi^2}{9^n}$ diverges.

QUESTION 1

$$\sum_{n=1}^{\infty} a_n = A$$
$$\sum_{n=1}^{\infty} b_n = B$$
$$\sum_{n=1}^{\infty} C_n = C$$

A, B, C are constants.

Does $\sum_{n=1}^{\infty} (a_n + b_n + C_{n+1})$ exists? Evaluate.

$\Rightarrow \sum_{n=1}^{\infty} (a_n + b_n + C_{n+1})$ is a partial sum of convergent series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} C_n$.

From the theory of subsequence, every subsequence of X_n converges to X as $n \rightarrow \infty$. to imply that the series X_n also converges to X.

Thus $\sum_{n=1}^{\infty} (a_n + b_n + C_{n+1})$ is also convergent.

We proceed to test for existence.

for a convergent series $\lim_{n \to \infty} X_n = 0$

But $\lim_{n \to \infty} \left\{ \sum_{n=1}^{\infty} (a_n + b_n + C_{n+1}) \right\} > 0$

thus the series exists not.

1)

QUESTION 2

Volume of a Sphere $= \frac{4}{3}\pi r^3$

Volume of one Sphere $= \frac{4}{3}\pi \cdot \left(\frac{1}{(\pi)^n}\right)^3 \quad \forall\, n = 1, 2, 3 \ldots$ Part

Total volume of the spheres $= \sum_{n=1}^{\infty} \frac{4}{3}\pi \cdot \left[\frac{1}{(\pi)^n}\right]^3$

3). We suppose $\sum_{k=1}^{\infty} x_k \neq \sum_{k=1}^{\infty} y_k$

$x_k \leq y_k \quad \forall\, k = 1, 2, \ldots$

$\sum_{k=1}^{\infty} x_k < \infty$ converges if $\sum_{k=1}^{\infty} y_k < \infty$ converges

Same applies to divergence.

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,\sqrt{n+1}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\leq \sum_{n=1}^{\infty} \frac{1}{n^{-2}}$

but from $\sum_{n=1}^{\infty} \frac{1}{n^p}$ we have $p = -2 \leq 1$.

thus $\frac{1}{\sqrt{n}}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,\sqrt{n+1}}$ also diverges

4) The height gainst after the 2^{nd} Collission

$= a(ah) = ah$

After 3^{rd} collision, h gain by rubber $= a(a)(ah) = ah$

and this goes on infinitely.

Adding all heights and the initial one we get S as

$S = h + ah + ah + ah + ---$

⇒ Summand of infinite geometric progression series with

$a = h$

$r = 1\alpha$

∴ total d ~~ta~~ is $h S = 1 - 9$ being expression of S in terms of h and α.

6). $\sum_{n=1}^{\infty} 1/a_n$ diverges.

<u>Proof:</u>

By definition:

$S_n(k) = \sum_{n=1}^{k} 1/n$ diverges.

$\iff \forall\, m > 0 \;\exists\, \bar{k}$ such that

$S_n(k) > m \quad \forall\, k > \bar{k}$.

$\Rightarrow S_{an}(k) = \frac{1}{a} S_n(k) = \sum_{n=1}^{k} \frac{1}{an}$ diverges since

$\forall\, m > 0 \;\exists\, \bar{k}$ such that $S_n(k) > aM \;\; \forall\, k > \bar{k} \Rightarrow$

$S_{an}(k) > M$.

Alternatively:

We can argue that since $\sum_{n=1}^{\infty} 1/n$ p-series diverges then $\frac{1}{an}$ ~~which~~ is divergent as a divergent series multiplied by a non-zero constant yields another divergent series.

9). $\sum_{n=1}^{\infty} (-1)^n n e^{-a}$

for an alternating series test :-

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + - - .$ for $a_n > 0$

will converge if.

a) $a_{n+1} \leq a_n$ and when $\lim_{n \to \infty} a_n = 0$.

$\sum_{n=1}^{\infty} (-1)^n ne^{-a}$

$a_n = ne^{-a} \Rightarrow n/e^a$

$a_{n+1} = \frac{n+1}{e^a}$

$a_{n+1} \geq a_n$ thus convergence . not satisfied.

$\sum_{n=1}^{\infty} (-1)^n n e^{-a}$ $\Rightarrow$ divergent.

7)

$$S_k = \sum_{k=1}^{\infty} (a_k - a_{k+1}).$$

Partial sums $\Rightarrow \{(a_1 + a_2 + a_3 - - - a_\infty) - (a_2 + a_3 + a_4 - - a_{\infty+1})\}$

$\Rightarrow \{a_1 + a_\infty - a_{\infty+1}\}$

$\Rightarrow a_1$